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Mathematical Programming Solutions In Constrained Optimal Control

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Abstract
Many economists are familiar with optimal control as a theoretical tool. However, the ability to empirically solve dynamic problems is limited. This paper introduces to the general Agricultural Economics audience a mathematical programming method which has been used success fully for over four years. The basic idea is a straightforward extension of static programming techniques already familiar to economists.

Disciplines
Agribusiness | Economic Theory | Other Mathematics | Set Theory

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Abstract

Dynamic optimization is difficult, especially with many limiting resources and a variety of constraints. Mathematical programming has been used successfully for over four years and promises to overcome these difficulties. Two solved examples are given: dynamic equilibrium in the long-run beef market, and the choice of discrete soil conserving technologies by a combination of free time optimal control problems.

Mathematical Programming Solutions in Constrained Optimal Control

Many economists are familiar with optimal control as a theoretical tool. However, the ability to empirically solve dynamic problems is limited. This paper introduces to the general Agricultural Economics audience a mathematical programming method which has been used successfully for over four years.[1] The basic idea is a straightforward extension of static programming techniques already familiar to economists.

Two approaches are common in dynamic optimization: 1) Dynamic Programming (DP), and 2) Linear Quadratic Gaussian Control (LQG). Each has its strengths, though judging by the number of empirical studies, DP seems to predominate. However, just as Achilles with his heel, each technique has its small, but mortal weakness.

An ideal optimal control algorithm would solve nonlinear control problems containing a large number of limiting resources and a mixture of equality and inequality constraints. It would calculate both the primal resource allocations over time and the dual values of those resources. Finally, the procedure would be routine with a single "canned" program for a variety of applications.

LQG is an elegant and refined algorithm of matrix multiplications. A description is found in Athens and solved examples in Dixon and Howitt, Noel, et. al, and Rausser and Howitt. Large control problems with any numbers of resources are routinely solved for both the primal and dual. Although the objective must be quadratic and the equations of motion linear, this is not overly restrictive for many applications. Unfortunately, LQG cannot admit inequalities, not even simple nonnegatively constraints. Hence it is best suited to "tracking" problems, minimizing the deviations from a known trajectory with control variables unrestricted in sign.

DP, on the other hand, is a brute force approach which enumerates feasible alternatives. A survey has been done by Kennedy, and a few recent examples are Burt, Yaron and Dinar, and Kilmer, et. al. DP has become more practicable as many of its weaknesses have been overcome. For example, DP solves only the primal. Of course, once the primal is known, estimates of dual variables can be obtained in a second step. A unique computer program was once required for each application, but now some computer codes exist with a degree of generality. Because a mixture of constraints can be intractible in pure DP, hybrid techniques link DP to linear or quadratic programming. However, the "curse of dimensionality" cannot be overcome. There will never be a DP solution to a problem with more than just a very few limiting resources.

We offer a third alternative for solving optimal control problems: Nonlinear Mathematical Programming (NMP). The potential of NMP for optimal control has been recognized in textbooks for some time. It is an elegant simplex type algorithm which, unlike LQG, will admit any objective and constraints which satisfy Kuhn-Tucker conditions. NMP solves the primal and dual simultaneously at a saddle point, is available in "canned" computer programs, and eliminates the need for hybrid techniques. Finally, NMP avoids DP's "curse of dimensionality" by evaluating the first order conditions of a problem instead of trying to enumerate the possibilities.

The reason NMP has remained a textbook example of optimal control is the huge constraint matrix for any problem with a reasonable time horizon. Fortunately the matrix is extremely sparse having very few nonzero elements. Advances in the field of operations research reduce by orders of magnitude the computation time and memory capacity required for a problem with sparse constraints. What were once considered difficult control problems are now easily solved. In fact, they are as easily solved as ad hoc multiperiod programming models.

In the remainder of the paper we give two examples of constrained control problems solved by NMP. Both are dynamic analogs of the kinds of problems routinely solved in static programming. The first example is an equilibrium model of the beef market. The second presents a method for optimally choosing between discrete soil conserving technologies. Theoretical derivations will be abbreviated to concentrate upon the solution method. These examples are early experiences with a new approach and by no means represent the full scope of possibilities for NMP.

Dynamic Long-Run Equilibrium in the Beef Market

Being normative, an optimal control problem can, in some cases, be a poor predictor of observed behavior. Instead, the dynamic optimum is important as a benchmark for comparison. Such is the case in the beef industry which has always been unstable in the long-run.

Briefly stated the problem of a representative beef producer who retains ownership from calving to slaughter is:

$$(1)\quad \text{Max } V_0 = \sum_{t=0}^{T-1} r^t \pi_t + r^T V_1$$

subject to:

$$S_{t+1} - S_t = I_t - L_t$$

$$I_t \leq \frac{\alpha}{2} S_{t-k};\ I_t \geq 0;\ L_t \leq S_t;\ L \geq \bar{I}_{t-d}$$

where V_0 is the net present value of beef production, V_1 is a terminal condition, π is the net revenue in each time period, S is the stock of beef cows, I is the investment in replacement heifers, L is the liquidation of beef cows, α is the reproduction rate with $\alpha/2$ being the heifers weaned, k is a production lag, d is the useful life of a cow, r is a discount factor, T is the terminal time, and t is a time subscript.

The equation of motion describes the change over time of the state variable for breeding stock, S. The control variables, I and L for investment and liquidation, are constrained by both lower and upper bounds. Investment is nonnegative, but cannot exceed the number of heifers available which is the number of heifers that were weaned and have matured by k years. Liquidation cannot exceed the total number of beef cows, but must meet a minimum culling rate of all cows which have outlived their useful life. Because of these constraints, a Kuhn-Tucker solution approach is necessary.

The terminal value, V_1, is the known sale value of all animals at time T. Net revenue, $\pi_t = P_t(\alpha S_{t-k} - I_t + r^d bL_t)$, where P is the net price and $\alpha S_{t-k} - I_t + r^d bL_t$ is the quantity of beef. In other words, the offspring produced, αS_{t-k}, less those diverted to investment, I_t, plus an allowance for cows liquidated, $r^d bL_t$, is the total beef marketed. Adjustment costs might also be considered in revenues. When the size of the breeding herd changes, other capital facilities such as feedlots and packing plants must change as well. Adjustment costs would allow inclusion of these effects without specifically modeling them.

With the addition of a net price equation, (1) can be aggregated into a market model. The net price equation is:

(2) $P_t = q(\alpha S_{t-k} - I_t + r^d bL_t) + Z_t$

where q is the negative slope multiplying the quantity marketed, and Z is an exogenous variable incorporating income, consumption of other meats, consumer habits, and so forth. The amount of non-fed beef, bL_t is depreciated by the factor r^d to make it's contribution to the market equivalent to that of fed beef, $\alpha S_{t-k} - I_t$.

With the inclusion of the price equation, the decision problem in (1) becomes a quadratic program. Each S, I, L, and P at a point in time is a separate primal variable. For the sake of exposition, assume the lag parameters, d and k, both equal 1 and choose a terminal time of

$T = 2$. Table 1 shows the primal/dual tableau of the beef market control problem.

The 15 primal constraints in Table 1 read from left to right after multiplying the constraint matrix by the vector of primal variables and comparing to the right-hand side as in the matrix equation $Ax \leq b$. Primal constraints 1 and 2 are the change in the stock of beef cows as the difference between investment and liquidation. Constraints 3 and 4 define the initial conditions on the stock of cows. Constraints 5, 6, and 7 are net price equations for each time period. Alternatively, net prices could be substituted into the objective, V_0, and the price variables eliminated from the problem.

Constraints 8 and 9, and constraints 10 and 11 are the upper and lower bounds on investment while 12 and 13, and 14 and 15 are upper and lower bounds on liquidation. The lower bound on investment is included for completeness, although it is a nonnegativity constraint imposed automatically by most mathematical programming algorithms.

The matrix notation for the dual equations is $c' = y'A$. Reading the tableau in Table 1 from top to bottom, c' corresponds to $[\partial V_o/\partial S_T, \ldots, \partial V_o/\partial S_{-1}]$, y' to $[\lambda_T, \ldots, \bar{\nu}_o]$, and A to the constraint matrix. It is easiest to associate the first dual variable, λ_T, with the first primal constraint in the first row of the constraint matrix, the second dual variable, λ_1, with the second row, and so on, until the last dual variable, $\bar{\nu}_o$, is associated with the fifteenth row. Then each of the 11 columns will contain a dual equation.

Table 1. Primal/dual Tableau for the Beef Market Example

$$\left[\frac{\partial V_0}{\partial S_T}\ \frac{\partial V_0}{\partial P_T}\ \frac{\partial V_0}{\partial S_1}\ \frac{\partial V_0}{\partial P_1}\ \frac{\partial V_0}{\partial I_1}\ \frac{\partial V_0}{\partial L_1}\ \frac{\partial V_0}{\partial S_0}\ \frac{\partial V_0}{\partial P_0}\ \frac{\partial V_0}{\partial I_0}\ \frac{\partial V_0}{\partial L_0}\ \frac{\partial V_0}{\partial S_{-1}}\right]$$

$$=$$

$$\left[\lambda_T\ \ \lambda_1\ \ \lambda_0\ \ \lambda_{-1}\ \ \varepsilon_T\ \ \varepsilon_1\ \ \varepsilon_0\ \ \tilde{\mu}_1\ \ \tilde{\mu}_0\ \ \bar{\mu}_1\ \ \bar{\mu}_0\ \ \tilde{\nu}_1\ \ \tilde{\nu}_0\ \ \bar{\nu}_1\ \ \bar{\nu}_0\right]$$

$$\begin{array}{r}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\ \\\end{array}
\begin{array}{c}\left[\begin{array}{ccccccccccc}
1 & . & -1 & . & -1 & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & -1 & . & -1 & 1 & . \\
. & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 \\
. & 1 & -q\alpha & . & . & . & . & . & . & . & . \\
. & . & . & 1 & q & -qr^d b & -q\alpha & . & . & . & . \\
. & . & . & . & . & . & . & 1 & q & -qr^d b & -q\alpha \\
. & . & . & . & 1 & . & -\alpha/2 & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & -\alpha/2 \\
. & . & . & . & -1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & -1 & . & . \\
. & . & -1 & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & -1 & . & . & 1 & . \\
. & . & . & . & . & -1 & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & -1 & .
\end{array}\right]\\
\begin{array}{ccccccccccc}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\end{array}\end{array}
\begin{bmatrix} S_T \\ P_T \\ S_1 \\ P_1 \\ I_1 \\ L_1 \\ S_0 \\ P_0 \\ I_0 \\ L_0 \\ S_{-1} \end{bmatrix}
\begin{array}{c} = \\ \leq \end{array}
\begin{bmatrix} 0 \\ 0 \\ \bar{S}_0 \\ \bar{S}_{-1} \\ Z_T \\ Z_1 \\ Z_0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \bar{I}_{-1} \end{bmatrix}$$

The constraint matrix of coefficients is 15 x 11. The vector $[S_T,\ldots,S_{-1}]'$ contains the 11 primal variables, and the vector $[\lambda_T,\ldots,\bar{\nu}_0]$ the 15 dual variables. Vector $[\partial V_0/\partial S_T,\ldots,\partial V_0/\partial S_{-1}]$ contains the 11 derivatives of the objective with respect to the primal variables. Vectors $[0,\ldots,Z_0]'$ and $[0,\ldots,\bar{I}_{-1}]'$ contain the 15 exogenous variables and initial conditions.

For example the terminal costate, λ_T, is determined by dual equation 1 as $\partial V_0/\partial S_T=\lambda_T$. The changes over time of the costate variables are defined in dual equations 3 and 7, while the initial costate is in dual equation 11. These and other dual equations are simply the first order conditions which result from differentiating a Lagrangian or Hamiltonian with respect to each of the primal variables.

Note that all nonnegativity and other constraints are specified in Table 1. If some constraints are excluded, the first order conditions are expressed in terms of complementary slackness. A textbook example might be, Max f(x) subject to $x>0$, giving slackness conditions $f' \leq 0$ and $xf' = 0$. However, there exists some slack variable, σ, so that $f'+\sigma=0$. Lets also be a slack variable with $x-s=0$. Then $f(x)=f(s) = f(x) + \sigma(x-s)$, and $\partial f/\partial s = f'=-\sigma$, or $f'+\sigma=0$. This is precisely the first order condition which results by setting the derivative of the Lagrangian, $f(x)+\sigma x$, with respect to x equal to zero. Complementary slackness has been shifted to the dual variable, $\sigma \geq 0$ and $\sigma x=0$. With the slack taken up by all possible dual variables, the first order conditions become equalities in Table 1.

In the parlance of mathematical programming, the vector of derivatives, $[\partial V_0/\partial S_T, \ldots, \partial V_0/\partial S_{-1}]$ is the c_j's with the remainder of each dual equation being the z_j's. At each iteration the pricing operation in a simplex algorithm selects the $z_j - c_j$ (dual equation or first order condition) which is farthest from being met. The solution is updated to satisfy the selected dual equation without violating any primal constraints. An optimal solution for the tableau in Table 1 will satisfy all first order conditions and solve the constrained optimal control problem in (1).

Notice the solution will be for a monopolist in the beef market because the decision maker is allowed to consider the derivatives $\partial V_0/\partial P_t = \varepsilon_t$ in dual equations 2, 4, and 8. If the NMP algorithm at hand allows complete control over the derivatives of the objective, an easy way to enforce a competitive equilibrium is to simply let $\partial V_0/\partial P_t = 0$. However, if the problem is solved by quadratic programming, or if the net price equation is substituted into the objective and prices eliminated as variables, this simple approach is no longer possible. Rather the trick of replacing the slope, q, of the net price equation by q/2 is required. Take the decision for investment at time 1 in dual equation 5, for example. Since $\partial V_0/\partial I_1 = (\partial V_0/\partial P_1)(\partial P_1/\partial I_1) = -\varepsilon_1 q$ the first order condition can be written $0 = -\lambda_T + 2\varepsilon_1 q + \tilde{\mu}_1 - \bar{\mu}_1$ which is tricked into the competitive solution by halving q.

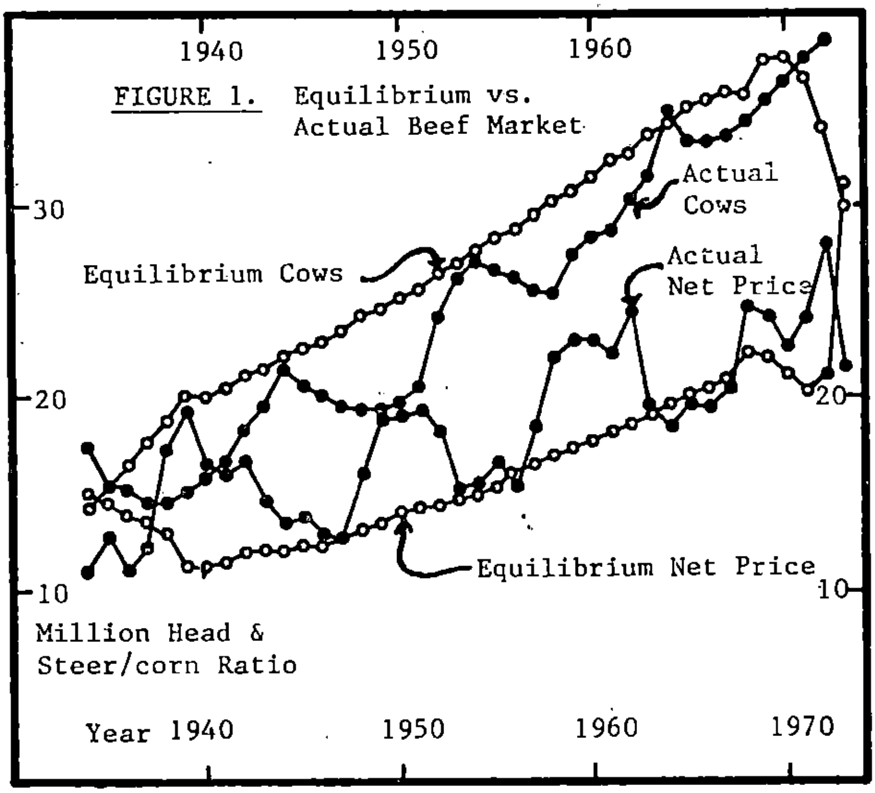
Of course the tableau of Table 1 with only three time periods could be solved by any quadratic programming package. The difficulty arises when longer time horizons are considered. The constraint matrix becomes extremely large very quickly. For example, the beef market model was expanded to a time horizon of 40 years from 1934 through 1973, estimated, and solved. The only binding inequality constraints were the lower bounds on liquidations. All others were eliminated. Still the constraint matrix consisted of 121 rows and 159 columns for a total of 19,239 elements. However, only 393 of these were nonzero. The nonlinear programming package, MINOS by Murtagh and Saunders, is specifically designed to handle sparse constraint matrices and easily reached the optimal solution.

Figure 1 shows the optimal level of stock and the net price for the competitive beef market as compared to the observed levels. As was mentioned, the optimal solution serves only as a benchmark for comparison. The explanation for the observed behavior of the beef market requires further investigation and the interested reader is referred to Hertzler.

Time to Adoption of Discrete Soil Conserving Technologies

Soil conservation is a widely researched topic with diverse approaches. Our example focuses upon one of the most conceptually difficult aspects: the farmer's adoption of discrete, indivisible tillage technologies. A chicken and egg question (two point boundary value problem) arises. The choice of a tillage system will depend upon relative costs and returns with one of those costs being the user cost of soil erosion. The unit costs of erosion are costates which can't be computed until the rate of erosion at each point in time is determined by the adoption of a tillage system. Hence the need for optimal control.

A common model in resource economics operates a single technology on one of many independent stocks. The farmer, however, operates one of many independent tillage systems on a single stock of soil. The control variables are no longer piecewise continuous. Rather the same technology is chosen for a wide range of user costs and then a switch occurs. Pontryagin's Maximum Principle does not apply in its usual form and must be extended (Michel).


1940
1950
1960
FIGURE 1. Equilibrium vs. Actual Beef Market
30
20
10
Actual Cows
Equilibrium Cows
Actual Net Price
Equilibrium Net Price
20
10
Million Head & Steer/corn Ratio
Year 1940
1950
1960
1970

For simplicity of exposition, suppose a farmer has only two alternatives for tilling the soil: fall moldboard plowing (FP), and spring disking (SD). Each may produce different yields and incur different costs under the same soil and fertilizer conditions. In particular, the more erosive technology, FP, must pay higher user costs for depleting soil resources. The length of time for which SD is selected can be expressed as the free-time optimal control problem

(3) $$\text{Max } V_0 = \sum_{t=0}^{t_1-1} r^t \pi_{SDt} + r^{t_1} V_1$$

subject to:

$$S_{t+1} - S_t = a_{SD}$$

$$N_{t+1} - N_t = -a_{SD} (N_t b_{SD}) + F_t$$

where V_0 is the net present value of farming, V_1 is a terminal condition, π_{SD} is the net revenue in each time period from using SD, S is the depth of soil lost to erosion, N is the stock of nutrients, F is the application of fertilizer, a_{SD} is the amount of soil lost in each time period using SD, b_{SD} is the proportion of the N nutrients left in the very upper layer by SD and susceptible to erosion, r is a discount factor, t_1 is the time to switch, and t is a time subscript.

The net revenue function, $\pi_{SDt} = (p_t - w_{yt})\, y_{SDt} - C_{SDt} - w_{Ft} F_t$ where p is the output price, w_y are yield dependent costs including the replacement of nutrients extracted by the yield, y_{SD} is yield for SD,

C_{SD} are fixed costs for SD, w_f is the price of fertilizer. The yield is determined by a generalized Leontief function, $y_{SD} = \min\ [y_P\ (S,N), y_N\ (S,N)]$ where y_P is plateau yield as a function of soil erosion and nutrients, and y_N is potential yield for the nutrients available.

The trick to solving a free time optimal control problem is to transform it into a fixed time problem through the introduction of a dummy control variable constrained to be between 0 and 1 (Robson). Rewrite V_0 in Lagrangian form as

$$(4)\quad V_0 = \sum_{t=0}^{T-1} [r^t \Phi_{SDt} \pi_{SDt} + \lambda_{t+1}(\Phi_{SDt} a_{SD} - S_{t+1} + S_t) + \psi_{t+1}(-\Phi_{SDt} a_{SD} N_t b_{SD} + F_t - N_{t+1} + N_t) + \bar{\mu}_t \Phi_{SDt} + \tilde{\mu}_t (1-\Phi_{SDt})] + r^{(\Sigma\Phi_{SDt})} V_1$$

where Φ_{SD} is a dummy control variable for SD with $\Sigma\Phi_{SDt} = t_1$, λ is the costate on soil erosion, ψ is the costate on nutrients, $\bar{u}$ and $\tilde{u}$ are Lagrange multipliers and T is a fixed time at least as large as t_1.

The switching condition is the unique aspect of choosing a discrete technology. From (3), the farmer will operate SD if $r^t \pi_{SDt} + r^{t+1}\ V_1 + r^{t+1}\ ((\partial V_1/\partial S_t) a_{SD} - (\partial V_1/\partial N_t)\ a_{SD} N_t b_{SD}) \geq r^t V_1$. Equivalently from (4) the farmer will set Φ_{SD} equal to one so long as $r^t \pi_{SDt} + \lambda_{t+1} a_{SD} - \psi_{t+1} a_{SD} N_t b_{SD} + \bar{\mu}_t - \tilde{\mu}_t \geq 0$. In words, $\lambda_{t+1} a_{SD}$ and $r^{t+1}(\partial V_1/\partial S_t) a_{SD}$ are the unit cost of soil erosion times the amount lost, $-\psi_{t+1}\ a_{SD} N_t b_{SD}$ and $-r^{t+1}\ (\partial V_1/\partial N_t) a_{SD} N_t b_{SD}$ are the unit cost of nutrient loss times the amount lost, and $\bar{\mu}_t - \tilde{\mu}_t$ and $-(r^t - r^{t+1}) V_1$ are the cost of delaying alternative V_1. Net revenue, $r^t \pi_{SDt}$, must at least cover these user costs of erosion and the cost of delay to continue operating SD.

In a free time problem, the magnitude of terminal value V_1 is also important, not just its derivatives as in a fixed time problem. Unfortunately V_1 is unknown at the time V_0 must be solved. In fact V_1 is itself a control problem for FP whose initial conditions depend upon the outcome of V_0. The choice of FP is defined analogously,

$$(5)\quad V_1 = \sum_{t=0}^{T-1} [r^t \Phi_{FPt}\pi_{FPt} + \lambda_{t+1}(\Phi_{FPt}a_{FP}-S_{t+1}+S_t) + \psi_{t+1}(-\Phi_{FPt}a_{FP}N_t b_{FP} + F_t - N_{t+1} + N_t) + \bar{\mu}_t \Phi_{FPt} + \tilde{\mu}_t\,(1-\Phi_{FPt})] + r^{(\Sigma\phi_{SDt}+\Sigma\Phi_{FPt})} V_2.$$

As before, V_2 may be a control problem which has V_3, also a control problem, as its terminal condition. Eventually one of the terminal conditions must be assumed known. In fact, in most economic problems, a terminal condition is simply a control problem the investigator chooses not to solve. Of course its always possible to let the discount term approach zero in the distant future.

A further difficulty now becomes apparent. Because the user costs of erosion are unknown, it is impossible to know, a priori, whether SD or FP is the initially preferred tillage system. Suppose FP is preferred in (4) and (5). Then all $\Phi_{SD} = 0$ and $V_0 = V_1$. Or suppose SD is initially preferred but there is the possibility of switching from SD to FP and reswitching from FP back to SD. In both cases, control problem V_2 must describe technology SD, and it would be prudent to let

V_3 describe FP as well. When more than two technologies are considered, this recursive linking of control problems becomes even more difficult.

Recursion as solved by Dynamic Programming (DP) is equivalent to the summation used in Nonlinear Mathematical Programming (NMP). The recursive definition for an example without discounting is $V_n = \Sigma_{t=t_n}^{t_{n+1}-1} \pi_{nt} + V_{n+1}$ for the n = 0, ..., N control problems. Assuming V_N is constant, the summation form is found by backward substitution and $V_0 = \Sigma_{n=0}^{N-1} \Sigma_{t=t_n}^{t_{n+1}-1} \pi_{nt} + V_N$. The t_n are variables to be chosen anywhere in the interval [0,T] because the n technologies can occur at any time in any order. Using dummy variables, the objective becomes $V_0 = \Sigma_{n=0}^{N-1} \Sigma_{t=0}^{T-1} \Phi_{nt} \pi_{nt} + V_N = \Sigma_{t=0}^{T-1} \Sigma_{n=o}^{N-1} \Phi_{nt} \pi_{nt} + V_N$. Now the SD and FP tillage systems in (4) and (5) can be combined into

$$(6)\quad \text{Max } V_0 = \sum_{t=0}^{T-1} [r^t(\Phi_{SDt}\pi_{SDt} + \Phi_{FP}\pi_{FPt})] + r^{(\Sigma\Phi_{SDt}+\Sigma\Phi_{FPt})} V_2$$

subject to

$$S_{t+1} - S_t = \Phi_{SDt}a_{SD} + \Phi_{FPt}a_{FP}$$

$$N_{t+1} - N_t = -\Phi_{SDt}a_{SD}(N_t b_{SD}) - \Phi_{FPt}a_{FP}(N_t b_{FP}) + F_t$$

$$\Phi_{SDt} \geq 0;\ \Phi_{FPt} \geq 0;\ \Phi_{SDt} + \Phi_{FPt} \leq 1.$$

In each time period, switching criteria for both SD and FP will be evaluated and any complex switching/reswitching behavior will be determined automatically. It should be noted, however, an integer programming approach is necessary if only one of Φ_{SD} or Φ_{FP} are allowed

Table 2. Primal/dual Tableau for the Soil Conservation Example.

$$\begin{bmatrix} \frac{\partial V_0}{\partial S_T} & \frac{\partial V_0}{\partial N_T} & \frac{\partial V_0}{\partial S_1} & \frac{\partial V_0}{\partial N_1} & \frac{\partial V_0}{\partial \Phi_{SD1}} & \frac{\partial V_0}{\partial \Phi_{FP1}} & \frac{\partial V_0}{\partial F_1} & \frac{\partial V_0}{\partial S_0} & \frac{\partial V_0}{\partial N_0} & \frac{\partial V_0}{\partial \Phi_{SD0}} & \frac{\partial V_0}{\partial \Phi_{FP0}} & \frac{\partial V_0}{\partial F_0} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \lambda_T & \lambda_1 & \lambda_0 & \psi_T & \psi_1 & \psi_0 & \bar{\mu}_{SD1} & \bar{\mu}_{FP1} & \bar{\mu}_{SD0} & \bar{\mu}_{FP0} & \tilde{\mu}_1 & \tilde{\mu}_0 \end{bmatrix}$$

$$\begin{array}{rc}
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{matrix} &
\begin{bmatrix}
1 & \cdot & -1 & \cdot & -a_{SD} & -a_{FP} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & -a_{SD} & -a_{FP} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \frac{-\partial N_T}{\partial N_1} & \frac{-\partial N_T}{\partial \Phi_{SD1}} & \frac{-\partial N_T}{\partial \Phi_{FD1}} & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \frac{-\partial N_1}{\partial N_0} & \frac{-\partial N_1}{\partial \Phi_{SD0}} & \frac{-\partial N_1}{\partial \Phi_{FP0}} & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot
\end{bmatrix} \\
& \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{matrix}
\end{array}
\begin{bmatrix} S_T \\ N_T \\ S_1 \\ N_1 \\ \Phi_{SD1} \\ \Phi_{FP1} \\ F_1 \\ S_0 \\ N_0 \\ \Phi_{SD0} \\ \Phi_{FP0} \\ F_0 \end{bmatrix}
\begin{matrix} \\ \\ \\ = \\ \\ \\ \\ \\ \le \\ \\ \\ \\ \end{matrix}
\begin{bmatrix} 0 \\ 0 \\ \bar{S}_0 \\ 0 \\ 0 \\ \bar{N}_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The constraint matrix of coefficients is 12 x 12. The vector $[S_T,\ldots,F_0]'$ contains the 12 primal variables and vector $[\lambda_T,\ldots,\tilde{\mu}_0]$, 12 dual variables. Vector $[-\partial V_0/\partial S_T,\ldots,\partial V_0/\partial F_0]$ contains the 12 derivatives of the objective with respect to the primal variables. Vectors $[0,\ldots,\bar{N}_0]'$ and $[0,\ldots,1]'$ contain the 12 exogenous variables and initial conditions.

to be positive in each time period. Otherwise, at the time of switching both can be nonzero.

For exposition, assume the farmer's time horizon, T, is 2 years. The primal/dual tableau of (6) is shown in Table 2. Interpretation of the tableau is analogous to that of the previous beef market example. The primal constraints contained in the rows are read from left to right, and the dual equations in the columns from top to bottom. The pricing operation of the simplex algorithm will select the dual equation which is most violated, and change the basis until all are satisfied.

One difference in the soil erosion example is the nonlinear equation of motion for nutrients, N. The notation within the constraint matrix, $\partial N_{t+1}/\partial \Phi_{SDt}$ signifies the derivative with respect to Φ_{SDt} of the equation of motion for nutrients and equals $-a_{SD}N_t b_{SD}$. Just as the derivatives of the objective must be re-evaluated at each iteration, now so must the coefficients of the constraint matrix.

The optimal switching conditions are dual equations 5, 6, 10, and 11. Noting that $\partial V_0/\partial \Phi_{SDt} = r^t \pi_{SDt}$, dual equations 5 and 10, are exactly the conditions discussed previously for the problem in (4). The time paths of the costate variables are also defined in Table 2. Dual equation 1 is the terminal condition on λ. Dual equations 3 and 8 equate the changes in λ over time to the discounted marginal value products, $\partial V_0/\partial S_t$, just as in the first order conditions derived from a Hamiltonian or Lagrangian.

From the point of view of the owner, a terminal value is the sale price of the land. To the purchaser, it is a control problem which values the land's future productive potential. Assuming a competitive land market with perfect information, the land will change hands for exactly V_2 in (6). The first farmer cannot optimally choose technologies without simultaneously calculating the decisions of those who will follow. Although a positive rate of discount makes it possible to ignore everything beyond 500 years, this is roughly the lifespan of 10 farmers and V_2 must be calculated explicitly.

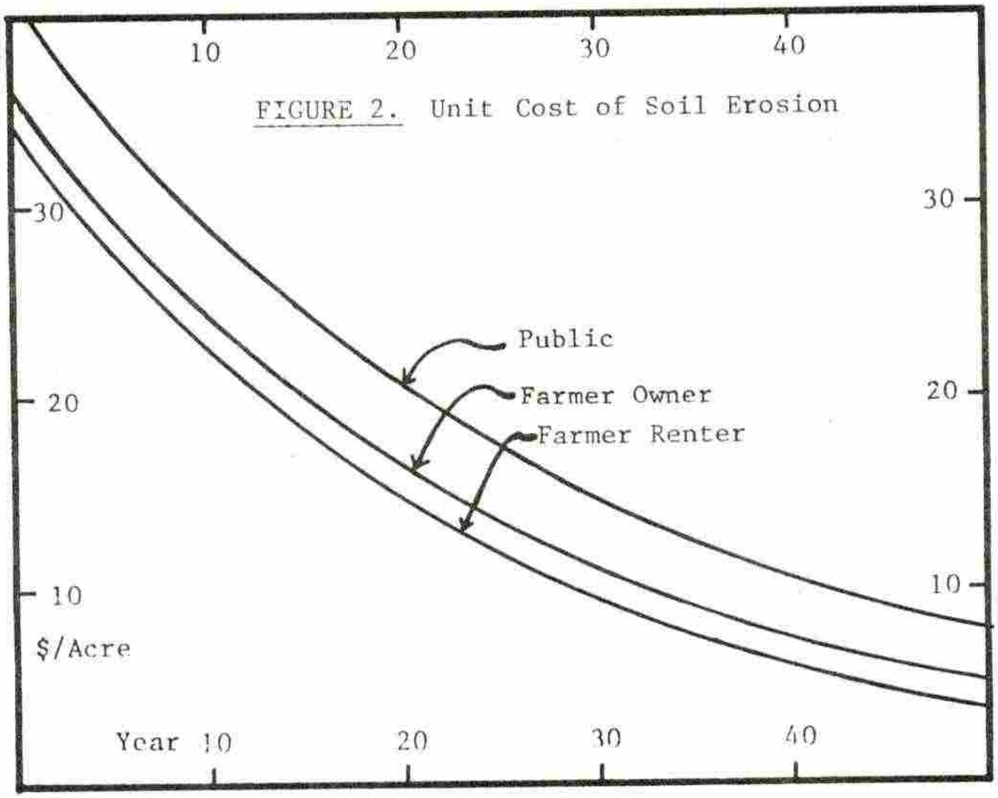
As actually solved, the tableau in Table 2 was expanded to 100 time periods. The first 50 periods were 1 year in length, the next 10 periods were 5 years in length, and the last 40, 10 years in length for a total of 500 years. Three technologies were considered, till planting (TP), spring disking (SD), and fall moldboard plowing (FP). The study area was in North Central Iowa with erosion rates calculated by the Universal Soil Loss Equation and yields of the generalized Leontief production function simulated by an agronomic model. A more detailed explanation is found in Ibanez-Meier.

Two nutrients, phosphorus and potassium have significant carryover from year to year. For 100 time periods, 200 nonlinear equations of motion would be required. While the MINOS algorithm can accommodate these equations of motion, simplifications are possible. In the generalized Leontief production function, optimizing farmers will apply phosphorus and potassium in predetermined proportions. Further, empirical price data suggested the yield plateau would always be optimal in the study area. Thus fertilizer levels could be computed directly without specific inclusion of the caryover effects. With these simplifications, the constraint matrix contained 201 rows and 401 columns for a possible 80601 elements of which 801 were nonzero. Solutions were easily obtained.

Figure 2 graphs the unit cost of soil erosion for three different land tenure situations. First is the public point of view with a 500 year time horizon and a 1% rate of time preference. Next is the farmer who owns the land and has a 4% rate of time preference. Finally is the renter with no resale value after 50 years and a 4% rate of time preference.

Because the social rate of time preference may embody an ethical judgement, reasonable people can disagree over the cost of soil erosion with no resolution. The interesting point is that even though user costs are different for the three tenure arrangements, all three till with SD. Less incentive to conserve does not imply greater rates of soil erosion in the discrete choice model. The most soil conserving technology, TP, was never adopted, even by the public. Further, since the cost of erosion is greatest in the present, the optimal time to adopt a tillage system is now or never. Only if future technological advances enhance TP should it be adopted in the study area.

The model's simple behavior of choosing conservation tillage now or never could have been anticipated because soil erosion favors present over future generations and so does a positive rate of time preference. But if the rate of time preference works in opposition to the rate of soil erosion, more complex switching behavior can result. For example, a negative rate of time preference would favor future


10
20
30
40
FIGURE 2. Unit Cost of Soil Erosion
30
30
Public
Farmer Owner
Farmer Renter
20
20
10
10
$/Acre
Year 10
20
30
40

generations and require greater conservation in the present, but less in the future. Expanding the model to include terracing, or reclamation of land might at times make the rate of soil erosion negative. Combined with a positive rate of time preference, complex switching behavior could result.

Finally, the intergenerational inconsistency of the discrete choice model must be acknowledged. The present farmer may choose a tillage system for his successors in the future which may not be the technology they actually adopt. The present farmer with a positive discount rate ignores anything beyond 500 years. However, a farmer 400 years hence will consider years 500 and beyond to be important and evaluate the switching conditions differently. Therefore the solution obtained should be interpreted as the decision of the current farmer only, given his best effort to calculate the sale value of land.

Summary

We have demonstrated the use of Nonlinear Mathematical Programming (NMP) in two examples of constrained optimal control. Our first example extends equilibrium analysis into a dynamic context with an application to the beef market. Our second example presents a solution method for a new area in optimal control, the choice of discrete technologies over time, and applies the method to soil conservation. The techniques of these examples can be modified for a wide variety of other problems.

We introduced primal/dual tableaus in Tables 1 and 2 to set up our examples and to show how NMP explicitly satisfies the first order conditions of dynamic optimization. Until recently, control problems have been difficult or impossible to solve by NMP. Advances in sparse matrix techniques allowed the MINOS mathematical programming package (Murtagh and Saunders) to easily find the optimal solutions.

NMP overcomes the limitations of previous approaches to optimal control and, as a technique, is easily accessible to economists familiar with static optimization or multiperiod programming. The success of our early experiences indicate NMP using sparse matrix methods has the potential to solve much larger dynamic problems. Someday, truly large problems may be decomposed into subproblems and solved by exploiting the special structure of the constraint matrix.

However, the major contribution of NMP may be at the interface between application and theory. An example is the subject of risk. E-V analysis is popular because of the limitations of quadratic programming. NMP, on the other hand, directly satisfies the first order conditions of any well behaved theoretical specification, and may be the key in future advances in stochastic optimal control (Blume, et al.).

1/ This area of research began at the University of California, Davis. The authors wish to acknowledge the contributions of Richard Howitt in applied optimal control.

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